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**Transient Temperature Change Model**

*Summary*—

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**Appendix A Derivation of Transient Two-field Model**

# Background and Objectives

# Method and Material

The ice sheet temperature is modelled using a second-order pde with a body source term for the sun irradiance. On the top surface of the ice sheet, the temperature is determined by a set convection heat transfer coefficient and set environmental conditions

Model also underlying water body?

The following is not modelled:

* change in ice sheet thickness due to melting or freezing
* convection heat transfer on bottom surface of ice sheet

## Ice Sheet

The ice sheet is sketched in Figure 2.



Figure 2

The solid (the sensor casing) has been assumed to be of stainless steel and the fluid to be water. Physical properties (same for both sensor makes) are given in Table 2.

Table 2 Material parameters

|  |  |  |  |
| --- | --- | --- | --- |
| **variable** | **symbol** | **value** | **unit** |
| density of solid |  | 8000 | kg/m3 |
| density of fluid |  | 1000 | kg/m3 |
| heat capacity of solid |  | 460 | J/(kg K) |
| heat capacity of fluid |  | 4200 | J/(kg K) |
| thermal conductivty of solid |  | 14 | W/(m K) |
| thermal conductivity of fluid |  | 0,6 | W/(m K) |

Process parameter values (same for both makes) are given in Table 3. It could be argued that the two sensor geometries should have different process parameter values, but it is difficult to make a reasonable quantification at present. The effective conductivity, taking account of some fluid recirculation in the cavity is .

Table 3 Process parameters (htc means heat transfer coefficient)

|  |  |  |  |
| --- | --- | --- | --- |
| **variable** | **symbol** | **value** | **unit** |
| ambient temperature |  | 20 | °C |
| fluid bulk temperature |  | 100 | °C |
| fluid bulk–wall htc |  | 2000 | W/(m2 K) |
| fluid bulk–interface htc |  | 20 | W/(m2 K) |
| solid–ambient htc |  | 10 | W/(m2 K) |
| cavity fluid–solid htc |  | 20 | W/(m2 K) |
| recirculation factor |  | 2 | — |

## Model

The initial condition becomes

and the boundary conditions become

with the Biot number

The two-field model takes into account a solid field (index s), the solid metal shell of the sensor, and one fluid field (index f), the fluid that fills the cavity of the sensor shell. The end tip of the sensor is assumed to be insulated. Heat is transferred from the fluid bulk in the pipe on which the sensor is mounted to both solid and the fluid fields; heat is in turn transferred from the fluid field to the solid field, and from the solid field to the ambient.

The non-dimensional distance and temperature used in this model are defined as

respectively.

### Transient Formulation

A dimensional two-field model with one equation and one set of boundary conditions for each field was derived, see Appendix A. Non-dimensional variables were defined and the equations were cast into a non-dimensional form. The non-dimensional equations for the solid and fluid fields, respectively, then are

with and the initial conditions

and boundary conditions

### Steady-state Formulation

The steady-state formulation is obtained by removing the time-dependent terms. After simplification, the non-dimensional equations for the solid and fluid fields, respectively, are

with the boundary conditions

## Numerical Solution of the Two-field Model

Jupyter Notebook with Python 3 was used to implement the numerical integration of the two-field model. More specifically, odeint from scipy.integrate was used for the steady-state formulation, while various matrix routines from numpy was used for the transient formulation.

The steady-state formulation was rewritten by defining so as to have only first-order derivatives, thus facilitating the numerical integration:

For the transient formulation, discretization (using a Crank–Nicolson scheme) resulted in the matrix equation

Here is the coefficient matrix for the vector of at time step , is the coefficient matrix for the vector of at time step , and is the vector holding any constants that appear in the boundary conditions.

## Standard Fin Model

Fins are normally homogeneous bodies, which is not the case for the present problem. A fin model may still give an indication of the magnitude of the difference between the measured temperature and the actual bulk temperature.

For a two-dimensional rectangular fin with an insulated tip, the temperature along the fin is[[1]](#footnote-2)

where

where is the fin perimeter and is the fin cross-sectional area.

# Results

## Steady-state Bulk Temperature Correction

The non-dimensional temperature distribution in the two fields for the base case defined in sections 2.1 and 2.2 is shown in Figure 3 for the Phoenix sensor. For this case, . This agrees well with the result of the standard fin model, . The result for the wika sensor was qualitatively similar but with a significantly smaller heat loss as .

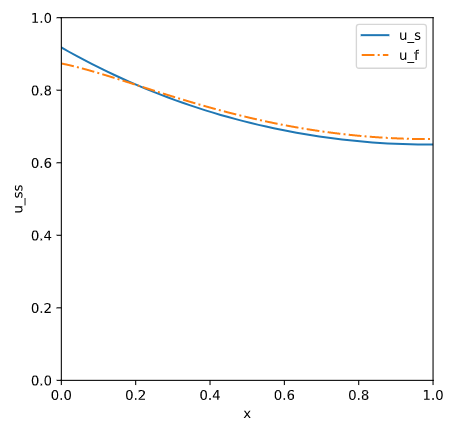


Figure 3 Steady-state solution of the non-dimensional temperature as a function of non-dimensional space (Phoenix geometry).

If the measured temperature is , then the correction needed to obtain the bulk temperature is

where . For the Phoenix sensor, the above result implies for the base case, having , that the measured temperature . The correction needed is thus 27 °C. For the wika sensor, the correction is 11 °C.

The sensitivity of (theoretical) correction needed for the tip temperature in the Phoenix case is shown in Figure 4.



Figure 4 Phoenix sensor sensitivity of temperature correction for a range of bulk and non-dimensional tip temperatures when the ambient temperature is 20 °C.

Corrections for a range of bulk temperatures and ambient temperatures in the Phoenix case are shown in Figure 5.



Figure 5 Phoenix temperature corrections needed for a range of bulk and ambient temperatures when .

## Response Time

The characteristic time used (Appendix A) is . Its numerical value for the present parameter values is for the Phoenix sensor and for the wika sensor.

The response time is somewhat arbitrarily defined as the time to reach 0,2 K from the steady-state solution after an imposed step change.

For a step change in fluid bulk temperature of 10 K, Figure 6, the response time for reaching the new steady-state (i.e. the time to reach 98 % of the step change) was 17 min for the Phoenix sensor and 8 min for the wika sensor. These times are of the same magnitude as their respective characteristic time. For the Phoenix sensor, 10 min was required to reach within 0,4 K (96 %).

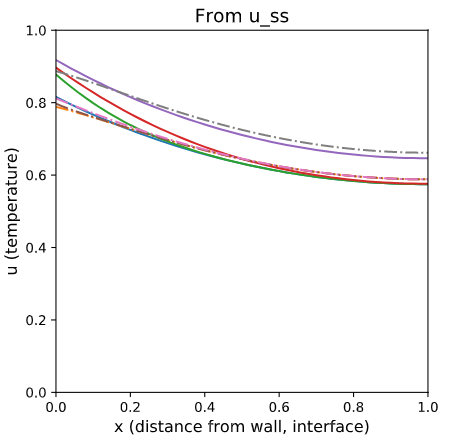


Figure 6 Phoenix sensor transients (red and green lines) for a sudden change from one bulk temperature (bottom pair of curves) to a second, higher, bulk temperature (top pair of curves).

# Discussion

Obviously, many simplifications have been made when modelling the sensor geometry. However, a sort of “worst case” route has been taken. For example, the contact resistances of stud attachment and the threads have not been considered.

Using the same values of the transport coefficients for the Phoenix and wika sensors is probably to the advantage of the wika sensor. Its smaller fluid cavity should probably result in lower transport rates.

## Steady-state Solution

Running the transient model for a “long” time should give the same result as the steady-state model and would be a check of the validity of the transient numerical method. This seems to have been the case as indicated in Figure 7.

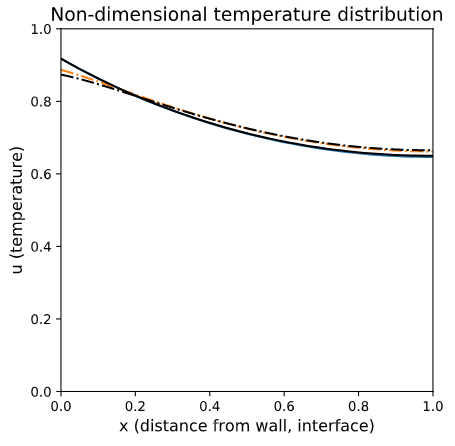


Figure 7 Comparison of long-time transient model solution to steady-state model solution (Phoenix geometry).

## Biot Numbers

The influence of the Biot numbers for the bulk–wall, bulk–cavity interface, solid–ambient, and fluid–solid heat transfers is shown in Figure 8 (Phoenix sensor). Each has been varied over a magnitude of 10 (except for due to numerical problems) with the other ’s held constant at their base case values (shown as filled circles). As expected, the heat transport over the solid–ambient interface is the most influential, and the situation gets worse, i.e. the read temperature becomes more different from the bulk temperature, as increases.



Figure 8 Influence of each of the Biot numbers in the steady-state model (Phoenix geometry).

## Means to Improve Accuracy

To maximise the steady-state bulk temperature accuracy, one needs to minimise the temperature correction, i.e. to (without changing the principal sensor topology)

* minimise the heat transfer rate to the ambient, e.g.
  + insulate the sensor
  + shorten the sensor
* maximise the internal fluid circulation, e.g.
  + increase the opening
  + introduce baffles to increase fluid exchange rate in the cavity
  + shorten the cavity

This will however lower the maximum bulk fluid temperature that can be measured safely at the sensor chip as this will be closer to the maximum allowed sensor temperature.

Another, obvious, means would of course be to move the temperature sensor into the fluid bulk, but that would imply a change of the sensor topology.

To minimise the response time, one needs to do the same as above.

# Conclusions

The conceptual measurement “error” for The Phoenix Sensors weps04 and wika mpr-1 have been estimated, as well as their response time.

The result is uncertain due to many simplifying assumptions in the models used. Nevertheless, the result indicates that the error can be significant at elevated temperatures: at 100 °C bulk temperature the reading may be 20–30 K lower.

The response time is estimated to 5–20 min, depending on sensor geometry.

It is desired to maximise accuracy and minimise response time. To maximise the steady-state bulk temperature accuracy, one needs to

* minimise the heat transfer rate to the ambient
* maximise the internal fluid circulation

To minimise the response time, one needs to do the same as above.

●

Appendix A  
Derivation of Transient Model

A material sheet is modelled as a flat sheet extending infinitely in the plane, but with finite thickness and a uni-directional transport of heat in the -direction. The sheet geometry, used for both the ice sheet and the water sheet below it, is shown below. The transport of heat is modelled by conduction only. When modelling the water sheet, convection is approximated using the eddy conductivity concept.



Figure A-1

**The Ice Sheet Model**

A differential heat balance for the ice temperature within the dashed control volume in the right-hand part of the above figure will be (no melting yet)

The first term on the left-hand side is the conduction into the control volume and the second is the short-wave radiative heat flux from the sun entering the control volume. The first term on the right-hand side is the conduction out of the control volume, the second is the short-wave radiative heat flux exiting the control volume, and the third is the accumulation of heat in the control volume (no melting yet).

Collect conduction and radiation terms, divide the equation by , and let . Then, assuming constant physical properties and transport coefficients and dividing by , one finds

The radiation heat flux from the sun (the irradiance) at an ice depth is[[2]](#footnote-3) , where is the absorption coefficient (in m–1). Hence, and (no melting yet)

The initial and boundary conditions for the ice sheet are

Here and

Note that , and that will be a part of the solution. The sun’s irradiation does not appear in the boundary condition as it is a source term in the governing equation.

The governing equations and the initial and boundary conditions are next recast into a non-dimensional form. For that, define

with and ???. The characteristic time scale is

The non-dimensional equation for the ice sheet then becomes

or, with ,

The initial condition becomes

and the boundary conditions become

with the Biot number

**The Water Sheet Model**

For the water sheet, the eddy conductivity concept is used and the assumption of constant conductivity does not hold. Hence,

with . is the real conductivity, here assumed constant, and is the eddy conductivity which depends on water convection situation (velocity).

The initial and boundary conditions for the water sheet are

Typically, (say).

**The Water–Ice Interface**

At the water–ice interface the temperature is 0 °C. The respective conductances of the water and the ice sheets imply the heat flux to and from the interface, respectively. Any difference (a non-zero net heat flux) will either melt ice or freeze water at the interface.

The selection of a water velocity, or an eddy conductivity, will determine the heat flux to the interface. The ice properties and the ice top surface conditions will determine the heat flux from the interface.

**Ice Sheet Bottom Melting and Freezing**

An ice sheet bottom melting rate can be computed once the ice and water sheet models have been integrated. However, the change in ice sheet thickness is not accounted for but is approximated to be constant.

**Ice Sheet Internal Porosity**

The change in porosity at every node is updated every time step.

When the ice sheet is heated (during the day), any temperature that is above 0 °C is recalculated into a melted amount, and the temperature is set to 0 °C.

When the ice sheet is cooled, if a node has a non-zero porosity and its temperature is below 0 °C, the temperature is recalculated into a re-freezed amount, and the temperature is set to 0 °C. However, if there is not enough porosity, the temperature is set to a corresponding value (somewhere between the initial temperature and 0 °C).

The interfacial node will introduce an artefact.

Appendix B  
Discretization of Transient Model

A Crank–Nicolson scheme is used. Hence,

Here is the coefficient matrix for the vector of at time step , is the coefficient matrix for the vector of at time step , and is the vector holding any constants that appear in the boundary conditions and any -independent source terms.

Starting with the differential equation

we write its corresponding discretization as

Define and collect time step terms on the left-hand side,

or, with an order of terms that agree better with the arrangement of arrays,

which corresponds to (but without the boundary conditions).

**Ice Sheet Boundary Conditions**

The boundary conditions are

At , we have (since the nodes are labelled ), or

Hence,

At , (since the nodes are labelled ), and it will be the ghost node that takes on the boundary condition, . Hence,

or? the last element takes on the boundary condition, , or

or

**Water Sheet Boundary Conditions**

The boundary conditions are

At , we have (since the nodes are labelled ), or

At , (since the nodes are labelled ), and it will be the ghost node that takes on the boundary condition, . Hence,

Appendix C

The porosity is defined as the volume non-ice (water, air, solids) to the total volume. At location :

The overall porosity is

1. Holman, J.P., *Heat Transfer*, section 2.9, McGraw-Hill (1989) [↑](#footnote-ref-2)
2. Ajne, M., *Ice Physics for Recreational Ice-users*, isbn 978 91 7575 162 7, p. 95ff [↑](#footnote-ref-3)